Invariant Kinematics on a One-Dimensional Lattice

in Noncommutative Geometry

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Abstract

In a one-dimensional lattice, the induced metric (from a noncommutative geometry calculation) breaks translation invariance. This leads to some inconsistencies among different spectator frames, in the observation of the hoppings of a test particle between lattice sites. To resolve the inconsistencies between the different spectator frames, we replace the test particle's bare mass by an effective locally dependent mass. This effective mass also depends on the lattice constant - i.e. it is a scale dependent variable (a "running" mass).

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We also develop an alternative approach based on a compensating potential. The induced potential between a spectator frame and the test particle is attractive on the average.

We then show that the entire formalism holds for a quantum particle represented by a wave function, just as it applies to the mechanics of a classical point particle.

1 Introduction

In [1, 2] it was found that the distances on a one dimensional lattice in a noncommutative geometry[1] differ from their classical analog. In noncommutative geometry the distances on a one dimensional lattice are given by

$$d_n = \begin{cases} 2\,a\,\sqrt{i(i+1)} & \text{, if } n = 2i \\[2mm] 2\,a\,(i+1) & \text{, if } n = 2i+1 \end{cases} \tag{1}$$

where $i \in \mathbb{N}$, $d_{-n} = d_n$, and a is the lattice constant, with units of length[2].

As can be seen directly, the distance to an odd n point has a unit anomaly, whereas the distance to an even n point is the geometric mean of the distances to the two nearby odd points. Moreover, we note that translation invariance is broken, since $d_n - d_{n-1} \neq d_{n+1} - d_n$. In the large n limit, however, translation invariance is restored asymptotically.

The fact that the distances given by (eq.1) thus break the lattice's translation invariance, represents a new phenomenon of a distance's dependence on reference frames. It follows directly from the basic fact that all the distances in (eq.1) were computed from some zero point, which the lattice sites labeling is been referred to. The selection of one of the lattice point as the zero point, was done without loss of generality. Therefore, *if several spectators are present on the lattice sites, with each spectator defining his location as the zero point[3] in*

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[1]Using a local discrete Wilson-Dirac operator

[2]The same eigenvalues for the distance operator were obtained in 2+1 dimensions, using the Ashtekar - Lewandowski formalism [4].

[3]Which is logical due to the *left-right* symmetry in his own world.

his own world, then due to the different sites' labeling by different spectators, the distance

between two lattice sites becomes reference frames dependent.

We may thus define the distance between the i-th and j-th sites, as been labeled in some

reference frame as the absolute value of the difference between the distances i.e.

$$d_{i,j} = \left| \frac{j}{|j|} d_j - \frac{i}{|i|} d_i \right| \quad \forall \ i, j \in \mathbb{Z} \tag{2}$$

We remind the reader of the reason for the introduction of a new definition of distance

in noncommutative geometry. It follows from the fact that the classical definition uses the

concept of a *path*, i.e.

$$d(a, b) = \inf \ l_\alpha (a, b) \tag{3}$$

where l_α is the length of the path α connecting points a and b. The infimum is taken over

all such path lengths. However, the concept of a path is only well defined on a smooth

manifold. One is thus forced to apply the noncommutative geometry definition, which is

more fundamental than the classical, and more general, since it fits all topological spaces.

This definition of a distance in noncommutative geometry is given by[3]:

$$d(a, b) = \sup_f \left\{ |f(a) - f(b)| \ : \ f \in A, \ \| \, [\, D, f \,] \, \| \leq 1 \right\} \tag{4}$$

where $a, b \in X$, $f \in A$, A is the algebra of functions on X, D is a Dirac operator (which is a

self adjoint operator with compact resolvent) acting in the Hilbert space H, and the norm on

the r.h.s is the norm of operators in H. Both definitions give the same result when the base

space is a Riemannian manifold. However the n.c.g definition has the advantage of being

applicable to discrete spaces too.

Noncommutative geometry, from its foundation was constructed in such a way that it fits the concepts used in Quantum Mechanics - mainly the passage from variables to operators. More details about the connection to Q.M. can be found in [3].

In that context the reader should not be surprised that we were led to the concept of reference-frames-dependent distances. Not with standing its novelty, what can be more natural in Q.M. than an influence of the observer on the results? We believe that a similar phenomenon will be revealed in a full theory of Quantum Gravity when this will be found.

In the following, we analyze some of the possible physical implications of the above result.

2   The model

Assume a one dimensional lattice with lattice constant a. On the lattice there are a *spectator* and a *test particle (t.p.)*. We assume that the spectator sticks to one site, which serves as the zero reference point in the spectator's system, from which distances are measured on the lattice. The t.p. has a mass μ and is situated at a site i. The dynamics of the t.p. on the lattice are characterized by a hopping motion from site to site on the lattice, where for each jump to occur there is some probability. Such dynamics are particularly appropriate in a scenario in which the system is a quantum system, in analogy with the motion of a particle between discrete energy levels. Note that the dynamics on the lattice can not be "smooth", i.e. the t.p. can jump between two lattice sites without "passing" through the

intermediate sites, as in jumps between Bohr's orbits for atomic electrons. Essentially, such hopping dynamics are known from solid state physics (e.g. Mott hopping conductance, also connected to percolation theory [5]). Quantum behavior is very much in the spirit of dynamics in a noncommutative geometry.

Two questions now arise. First, how do we treat *time*? The second question relates to the *probabilities* for the test particle's hops – how do we go about evaluating them?

2.1   The issue of time

As we shall see, there are three possible answers (and perhaps even more) for the first question. The first possible answer would claim that there might be no need at all for an answer, since one can deal with probabilities without involving time dependence, in the same way as one deals with the probabilities in dice, namely, without caring about the rate at which the dice are thrown. Thus, in this solution, the jumps' rate does not affect the probabilities. However, having in mind physical models and not just mathematical ones, we believe that the question cannot be eliminated. The second possible answer is that time is an extra dimension, as is usually done. Thus the total spacetime is $\mathbb{Z} \otimes \mathbb{R}$ or $\mathbb{Z} \otimes \mathbb{Z}$, depending on whether the time is continuous or discrete, respectively. But, since the results for the distances on a lattice which we use here are taken only for the one- dimensional case, and the distances on those two-dimensional spaces could be different from what we use, we prefer not to adopt this answer. Thus, although the second answer might be the correct one physically, we select for our purposes the following third answer, which enables us also to apply our result (1) for the distances. This third answer consists in using our quantum hops to define and measure time

intervals, in the spectator system. *To measure time, we count the t.p. jumps.* In this context, time is not an extra dimension, but rather a parameterization defined by counting, similar to the counting of the vibrations of the quartz in a digital watch. Although some vibrations might have longer periods (as measured by counting the vibrations of a more accurate watch) time can still be defined as the result of an exact counting of the vibrations. [4]

2.2   The probabilities issue

To determine the probabilities for the t.p.'s hops on the lattice, we first examine the propagator of a free boson on a translation-invariant space: $\frac{1}{p^2+\mu^2}$. The Fourier transform of this propagator is essentially the correlation function between different locations in the coordinate space. It seems reasonable to take the correlation function between two different sites as proportional to the probability of a jump between them: the more they are correlated, the stronger the chances for the t.p. to make that jump. Working with a one-dimensional lattice, we take the one-dimensional Fourier transform of the propagator. The probabilities will thereby be of the form:

$$P_{i,i+j} = \frac{e^{-\beta|x_i - x_{i+j}|}}{\beta f_i\left(\beta\right)} = \frac{e^{-\frac{\beta d_{i,j}}{a}}}{\beta f_i\left(\beta\right)} \tag{5}$$

where $\beta = \mu a$ (μ - is the t.p. mass, and a is the lattice constant), i and $i+j$ are the initial and the final sites of the t.p. relative to the spectator, $x_n = \frac{n}{|n|}\frac{d_n}{a}$, and $f_i\left(\beta\right)$ is the normalization of the probabilities. Remembering that the lattice translation invariance is broken and only

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[4]As one can see, with zero velocity not being allowed for a particle in the third option, one regains the Q.M. concept that the location and the momentum can not be simultaneously defined, as well as the classical concept that time can not be stopped.

reached asymptotically - we take the above expression for the probabilities only as a first guess, a form that they should roughly follow. The normalization is given by the following formula:

$$f_i\left(\beta\right) = \frac{1}{\beta} \sum_{j=-\infty}^{\infty} e^{-\beta|x_i - x_{i+j}|} \tag{6}$$

Note that for $i = 0$ the t.p. is initially situated at the same site as the spectator.

As can be immediately seen, if β is a constant, the hopping probabilities for the t.p. as seen in the spectator's frame differ from those evaluated in the t.p. frame. This is a direct outcome of the fact that $|x_i - x_{i+j}| \neq |x_j|$ where the l.h.s is the distance as observed in the spectator frame and the r.h.s is the distance as observed in the t.p. frame. In the following section, we discuss the meaning of this incompatibility between the probabilities in the two systems, and ways of resolving this difficulty.

3   Resolving the probability paradox

3.1   Further aspects of the incompatibility between the probabilities in the spectator and the t.p. frames.

A closer look at the distances involved will reveal that for a t.p. initially situated at an even site $i = 2l$, the most probable jump, as evaluated in the spectator frame is to the site $2l - 1$ (i.e. $j = -1$). If the t.p. was initially located at the site $i = 2l - 1$, the most probable jump, as evaluated in the spectator frame is to $2l$ (i.e. $j = +1$). However in the t.p. own frame, there is no difference in the probabilities for jumps to the left or to the right.

Another anomaly which is an outcome of the above incompatibility can be seen when inspecting distances between sites whose indices differ by two units. Though the distance between two neighboring odd points is always $2a$ (except for $i_1 = -1, i_2 = 1$), the distances between two neighboring even points form a decreasing series, tending to $2a$, when $n \to \infty$. In the spectator frame, the effective probability for the test particle's motion will thus be larger in an outward direction (relative to the spectator), i.e. the t.p. *drifts away*. In the t.p. frame, however, as there is no difference between its two sides, the t.p. should just stay in its initial position (on the average). Note, moreover, that different spectators - located each at a different lattice site (i.e. with different origins in their respective frames) - will observe different probability distributions.

Though the probability space in the t.p. frame is a commutative space (in the sense that when defining P_j as the probability of jumping j sites to the right, then $P_k P_j = P_j P_k$), it is noncommutative in the spectator frame, where $P_k P_j \neq P_j P_k$. This point summarizes the inconsistencies between the two frames.

Since the probabilities are thus not the same, whether between spectator and t.p. frames, or between two spectator frames, applying our "third option" as defined in section 2.1, we observe that *the time intervals are not synchronized*. We thus set upon removing the anomalies by an appropriate corrective mechanism.

3.2 Resolving the incompatibility: the frame-dependent mass solution

All above anomalies were obtained under the assumption that the mass - μ, of the t.p. is the same in both the spectator and the t.p. frames. Our solution is to abandon the assumption that the t.p. mass is a global parameter and assume instead that it is a local variable. In the following we demonstrate that this is a complete resolution of our paradoxes.

We require the probabilities $P_{i,i+j}$ and $P_{0,j}$ to be equal in both spectator and t.p. frames, i.e. using localized masses, we restore translation invariance for the probability of hopping $j-$sites.

$$\forall \, i,j \, \in \mathbb{Z} \; ; \qquad P_{i,i+j} = \frac{e^{-a\mu_{i,i+j}|x_{i+j}-x_i|}}{a\mu_{i,i+j}f\left(a\mu_0\right)} = \frac{e^{-a\mu_0|x_j|}}{a\mu_0 f\left(a\mu_0\right)} = P_{0,j} \tag{7}$$

where $\mu_{i,i+j}$ is the mass of the t.p. as observed in the spectator frame, while hopping from the i site to the $i+j$ site, and μ_0 is the mass in the t.p. system. The normalization constant $f\left(a\mu_0\right)$ is defined as:

$$f\left(a\mu_0\right) = \sum_{j=-\infty}^{\infty} \frac{e^{-a\mu_0|x_j|}}{a\mu_0} = \frac{2}{a\mu_0}\left(\frac{1}{2} + \frac{e^{-2a\mu_0}}{1-e^{-2a\mu_0}} + \sum_{j=1}^{\infty} e^{-2a\mu_0\sqrt{j(j+1)}}\right) \tag{8}$$

where we are using (1)[see figure 1 in the Appendix], thus normalizing the probabilities (i.e. $\sum_{j=-\infty}^{\infty} P_{0,j} = \sum_{j=-\infty}^{\infty} P_{i,i+j} = 1$).

From (7) we obtain the following relation between $\mu_{i,i+j}$ and μ_0:

$$a\left(\mu_0|x_j| - \mu_{i,i+j}|x_{i+j}-x_i|\right) = \ln\left(\frac{\mu_{i,i+j}}{\mu_0}\right) \tag{9}$$

As one can see, equation(9) can not be analytically solved for $\mu_{i,i+j}$, but if one assumes

for simplicity that:

$$a\mu_{i,i+j}\,|x_{i+j} - x_i| \ll 1 \; ; \quad a\mu_0\,|x_j| \ll 1 \tag{10}$$

one finds from (7) that the mass as seen by the spectator is given by

$$\mu_{i,i+j} \simeq \frac{\mu_0}{1 + a\mu_0\,[|x_{i+j} - x_i| - |x_j|]} \approx \mu_0\,[1 - a\mu_0\,(|x_{i+j} - x_i| - |x_j|)] \tag{11}$$

We emphasize that if one assigns Planck-length value l_p to the lattice constant a (i.e. $\sqrt{\frac{\hbar G}{c^3}}$), then the limit (10) should be interpreted as the low mass limit $\mu \ll m_p$, where m_p is a Planck-mass (i.e. l_p^{-1}). Thus, the mass formula (11) should be seen as semi-classical.

By setting the probabilities to be equal in the two frames, we are essentially synchronizing the relevant times - applying our definition of time intervals, based on counting jumps. The price, however, has been paid by the transformation of the mass μ_0 into a local and dynamical variable - $\mu_{i,i+j}$ (local and dynamical because of the i and the j dependence respectively).

There is a very rough analogy with the emergence of the Doppler effect in Special Relativity. However, whereas in the Doppler effect the frequency of a wave emitted by a system moving towards the spectator is blue-shifted, i.e. always increases in the spectator frame, in our case, the t.p. mass as observed in the spectator frame $\mu_{i,i+j}$ can be smaller than μ_0 , even when the t.p. jump is directed towards the spectator. The fact that the physical mass $\mu_{i,i+j}$ depends on the lattice constant a, aside from its dependence on μ_0 and on the location, suggests that $\mu_{i,i+j}$ depends on the scale. This behavior allows us to think of the mass as a "running mass", as happens in the context of the renormalization procedure in field theory (see figure 2 in the Appendix).

Adopting the above concept of a local mass, we have thus achieved full agreement between

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the spectator and t.p. probability values. There is thus also a synchronization of time measurements between the frames, and there is no preferred direction on the lattice.

3.3 The potential approach

Instead of modifying the kinematics, by constraining the mass to be a local variable, in our effort at restoring covariant dynamics for the t.p., we could use an alternative, "dynamical" approach, by introducing potentials. With the distances on the lattice not being translation invariant, the propagator of a boson particle situated at the $i-th$ site (in the spectator frame) could be written in terms of a potential $V_{i,}$,

$$\left[\, p^2 + \mu_0^2 - V_i\left(|x_{i+j} - x_i|\right)\,\right]^{-1} \tag{12}$$

where $V_i\left(|x_{i+j} - x_i|\right)$ is induced by the metric defined in (1) and (2). As we demonstrated, this is, on the average, a repulsive force,

$$F_{avg} \equiv -\left(\frac{\partial V}{\partial x}\right)_{avg} > 0 \tag{13}$$

To prevent this outward drift of the t.p. in the spectator frame, and to establish a correspondence between the t.p.'s propagation in all spectator frames, we then postulate *an induced attractive potential $V_i^{ind}\left(|x_{i+j} - x_i|\right)$ which will exactly cancel $V_i\left(|x_{i+j} - x_i|\right)$. This is somewhat reminiscent of Einstein's 1917 introduction of the cosmological constant, to stabilize the universal geometry. The propagator of the t.p. in all spectator frames are now the same as for a free particle $[p^2 + \mu^2]^{-1}$. The existence of the induced potential, in some aspects, is also similar to the Lentz law in classical electromagnetism.*

One can identify the modified mass in (7) with the following effective mass, defined as:

$$\mu_{i,i+j}^{eff} = \sqrt{\mu_0^2 + V_i^{ind}\left(|x_{i+j} - x_i|\right)} = \sqrt{\mu_0^2 - V_i\left(|x_{i+j} - x_i|\right)} \tag{14}$$

In other words, one can deduce from $\mu_{i,i+j}$ the value of the induced potential between the t.p. and the spectator (in the low mass limit (10) one can use (11) and (14)). Summarizing this approach, we note that though the distances break translation invariance and while the mass preserves its global definition, the hopping probabilities – and the related beats of the clock – have become translation invariant.

Summary

We have shown that the induced metric of a one-dimensional lattice, (from a noncommutative geometry calculation) breaks the lattice translation invariance and therefore also becomes reference-frame dependent, possibly leading to some paradoxes concerning the fundamental issue of general covariance. We have suggested a solution of the paradoxes, essentially at the probabilistic level - which in our opinion is the most important aspect, at least from the measurement aspects of Quantum Mechanics. It was done by either allowing an effective locally dependent mass which thus becomes a "running mass", or by introducing a compensating potential. Both ways lead to a commutative probabilistic space which is general covariant.

Acknowledgment

Appendix

A. The test particle as a wave function

Up to this point, we have treated the t.p. as a classical point particle. In the following, we go over to quantum mechanics - i.e. the t.p. is now a quantum particle, represented by a wave function - $\varphi(k)$ which is defined over the entire lattice.

Let $\varphi \in \mathcal{L}^2$ (i.e. a square integrable function) be normalized so that $\|\varphi\| = 1$. Let $(\psi_j)_{j \in \mathbb{Z}}$ be a complete set of orthogonal states which span the Hilbert space - $\mathbb{H}$ which is defined over the lattice. The Hilbert space basis $(\psi_j(k))_{j,k \in \mathbb{Z}}$ is defined as follows:

$$\psi_j(k) = \delta_{j,k} \tag{15}$$

where the j -index stands for a state in the Hilbert space, while the k -index stands for the lattice site. Any φ can now be represented by the following sum:

$$\varphi = \sum_{j=-\infty}^{\infty} \alpha_j \psi_j \tag{16}$$

where the α_j -s' are complex numbers. Since $\|\varphi\|^2 = 1$ it follows that $\sum_{j=-\infty}^{\infty} |\alpha_j|^2 = 1$.

As in Q.M. the $|\alpha_j|^2$ represent the probability for the particle to be found in a measurement at the j-site of the lattice. One can thus treat the $\|\varphi\|^2$ as a distribution, for which the $|\alpha_j|^2$ -s' form a probability space.

Let $\|\varphi\|_i^2$ be a distribution centered at the i site of the lattice. We define $(\alpha_j)_i$ as the coefficients in (16) for the $\|\varphi\|_i^2$. We observe the similarity between $|\alpha_j|_i^2$ and $P_{i,i+j}$; both define the probability for the t.p. to be initially at the i - site and after a measurement, in the $(i+j)$ - site. We have thus essentially showed that the t.p. can also be represented by

a wave function, with the effective mass (or the induced potential) having the same features as before.

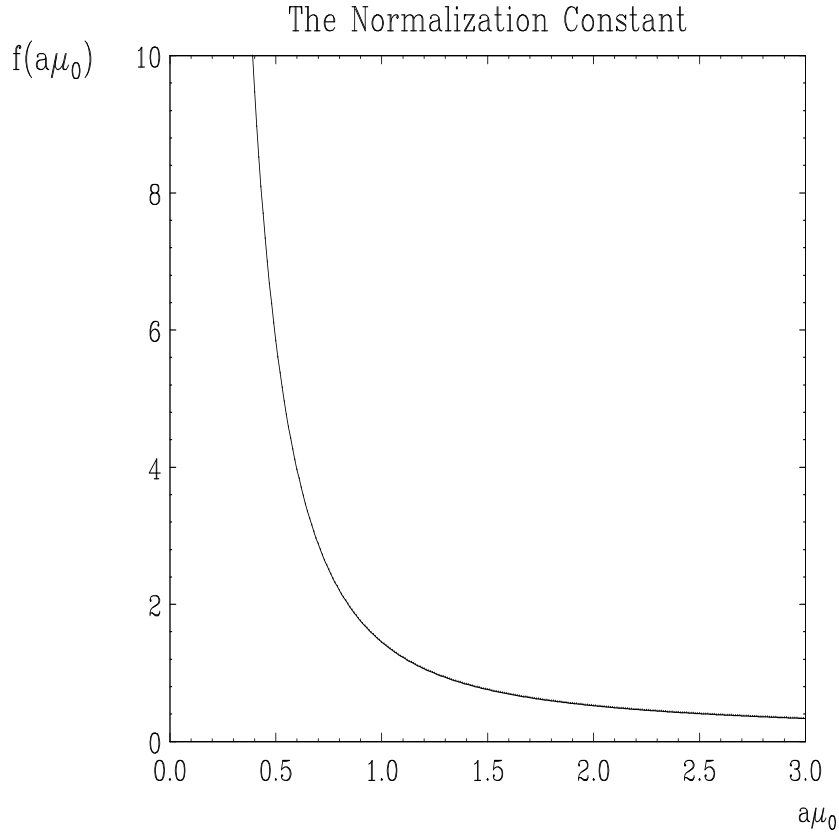## B. In the following we exemplify some of the formulae found in the article.

The Normalization Constant

$f(a\mu_0)$

[figure: plot of $f(a\mu_0)$ versus $a\mu_0$, a decreasing curve from high values near the left axis down toward about 0.4 at $a\mu_0 = 3.0$; vertical axis from 0 to 10, horizontal axis from 0.0 to 3.0]

$a\mu_0$

Figure 1.The dependence of the normalization constant given in (8) on

$a\mu_0$ (i.e. on the product of the lattice constant by the "bare" mass).

Figures 2a - 2e reproduce some examples of the running mass in (9). The i and the j indices represent the initial and the hopped-to sites of the test particle, respectively. μ_0 is the t.p. mass in its own frame, and a is the lattice constant.
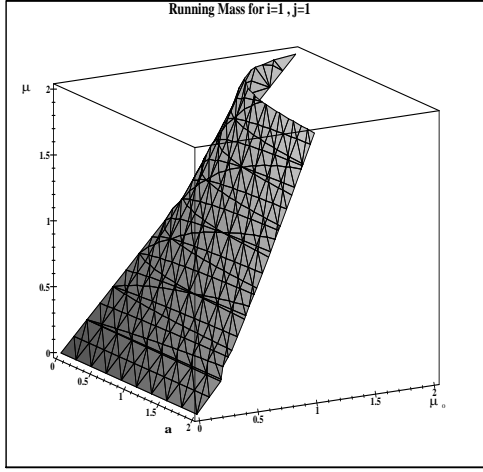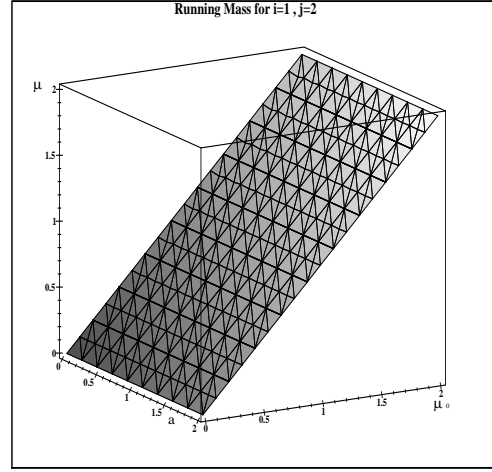
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Fig. 2a

Running Mass for i=1 , j=1

Fig. 2b

Running Mass for i=1 , j=2

Fig. 2c

Running Mass for i=1 , j=3

Fig. 2d

Running Mass for i=1 , j=4

Fig. 2e

Running Mass for i=1 , j=30

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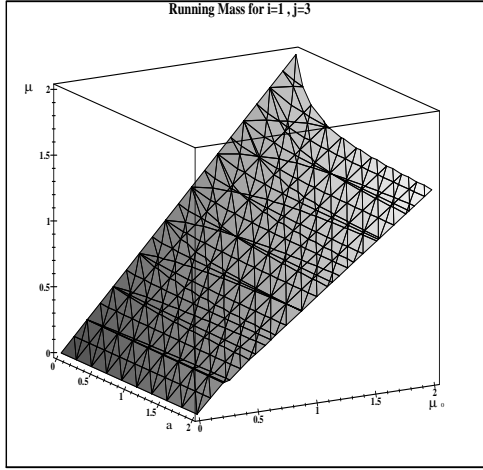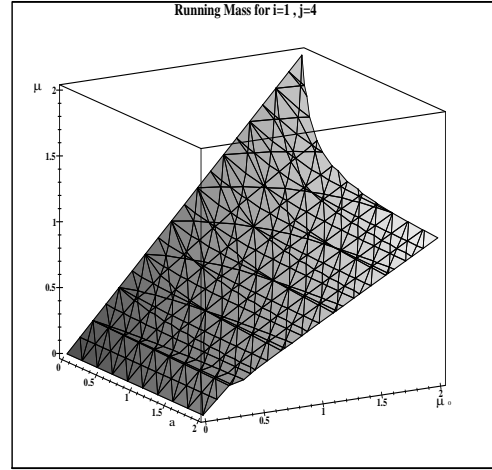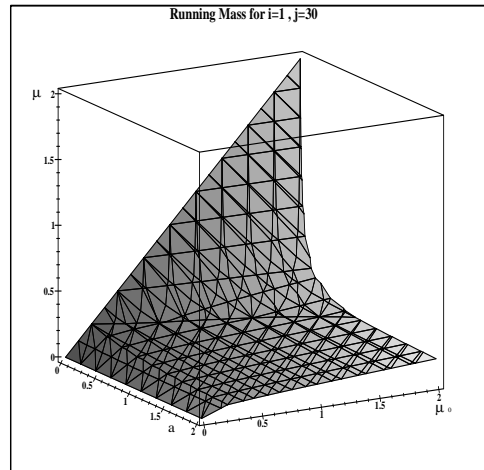
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